Inequality

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Let $a_1, a_2, \ldots, a_n$ be positive real numbers. Prove that

$$\prod_{k=1}^{n} \ln(1+a_k) \le \left(\ln\left(1+\sqrt[n]{\prod_{k=1}^{n} a_k}\right)\right)^n$$

Solution by Arkady Alt , San Jose, California, USA.

Let $f(x) := \ln(\ln(1+e^x))$. Since $1+e^x > 1$ for any $x \in \mathbb{R}$ then $Dom(f) = \mathbb{R}$.

Since $f''(x) = \dfrac{e^x(\ln(e^x+1) - e^x)}{(\ln^2(e^x+1))(e^x+1)^2} < 0$ for any $x \in \mathbb{R}$ then $f(x)$ is concave down in $\mathbb{R}$

and, therefore, for any real $x_1, x_2, \ldots, x_n$ by Jensen's Inequality

(**1**) $$\sum_{k=1}^{n} \ln(\ln(1+e^{x_k})) \le n \ln\left(\ln\left(1+e^{\frac{1}{n}\sum_{k=1}^{n} x_k}\right)\right).$$

By replacing $(x_1, x_2, \ldots, x_n)$ in (**1**) with $(\ln a_1, \ln a_2, \ldots, \ln a_n)$ we obtain

$$\sum_{k=1}^{n} \ln(\ln(1+a_k)) \le n \ln\left(\ln\left(1+\sqrt[n]{\prod_{k=1}^{n} a_k}\right)\right) \Leftrightarrow \prod_{k=1}^{n} \ln(1+a_k) \le \left(\ln\left(1+\sqrt[n]{\prod_{k=1}^{n} a_k}\right)\right)^n.$$